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3-branes on resolved conifold

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Abstract: The type-IIB supergravity solution describing a collection of regular and fractional D3-branes on the conifold (hep-th/0002159) was recently generalized to the case of the deformed conifold (hep-th/0007191). Here we present another generalization — when the conifold is replaced by the resolved conifold. This solution can be found in two different ways: (i) by first explicitly constructing the Ricci-flat Kähler metric on resolved conifold and then solving the supergravity equations for the D3-brane ansatz with constant dilaton and (self-dual) 3-form fluxes; (ii) by generalizing the "conifold" ansatz of hep-th/0002159 in a natural "asymmetric" way so that the 1-d action describing radial evolution still admits a superpotential and then solving the resulting 1-st order system. The superpotentials corresponding to the "resolved" and "deformed" cases turn out to have essentially the same simple structure. The solution for the resolved conifold case has the same asymptotic UV behaviour as in the conifold case, but unlike the deformed conifold case is still singular in the IR. The naked singularity is of repulson type and may have a brane resolution.

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Contents

1. Introduction

To construct supergravity (and string-theory) duals of less supersymmetric gauged theories one is interested in "$4+1+$ compact space" type backgrounds with extra p-form fluxes [1]. It is natural to try to generalize the original AdS/CFT correspondence [2] by considering D3-branes in more general transverse space backgrounds, e.g. placing them at conical singularities [3]–[6]. This idea has been developed further [7, 8] by adding "fractional" D3-branes (D5-branes wrapped over 2-cycles) [9], exploiting the fact that topologically the base space of the conifold ($\mathbb{T}^{1,1}$) is $S^2 \times S^3$. In [8] it was argued that the dual field theory should be a non-conformal $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ gauge theory and some key effects of introducing fractional branes were discussed, including breaking of conformal invariance and structure of the logarithmic RG flow.

The corresponding supergravity solution describing a collection of N D3-branes and M fractional D3-branes on the conifold was constructed in [10]. This solution has the standard D3-brane-type metric but with the "harmonic function" replaced by $h(r) = 1 + Q(r)/r^4$, $Q(r) = c_1 g_s N + c_2 (g_s M)^2 \ln r/r_0$. The logarithmic running of the "effective charge" $Q(r)$ implies the presence of a naked singularity at small r (in IR from dual gauge theory point of view). A remarkable way to avoid the IR singularity was found [11]: one is to replace the conifold by the deformed conifold keeping the same D3-brane structure of the 10-d metric and generalizing the 3-form ansatz appropriately.

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The deformed conifold solution [11] has the same large r (UV) asymptotic as the original conifold one [10] but is regular at small r, i.e. in the infrared. In [11] some desirable properties of this background were established, including the gravity counterparts of the existence of confinement and chiral symmetry breaking in the dual gauge theory.

It is of obvious interest to explore further the class of backgrounds which have similar "3-branes on conifold" type structure (potentially including also the solution of [15]). Given the topology of the base space, there are two natural ways of smoothing out the singularity at the apex of the conifold. One can substitute the apex by an S^3 (deformation) or by an S^2 (resolution). Here we complement the conifold [10] and deformed conifold [11] solutions by constructing explicitly the background corresponding to the resolved conifold case.[1] The type-IIB supergravity solution we find coincides with the original background of [10] for large r but has somewhat different (though still singular) small r (IR) behavior. The singularity of the analog of the solutions of [10, 11] in the resolved conifold case was anticipated in [16, 17].

One may discover this solution using two different strategies. One may start with the "conifold" ansatz for the 10-d background in [10] and generalize it in a very simple and natural way by allowing an "asymmetry" between the two S^2 parts in the metric and in the NS-NS 3-form introducing two new functions. Assuming the spherical symmetry as in [10] one can then obtain the resulting supergravity equations from a 1-d action describing evolution in radial direction. Remarkably, as in [10], the potential term in this action can be derived from a superpotential. This is true also in the "deformed" case of [11] and is consistent with expected $\mathcal{N} = 1$ supersymmetry of the resulting solution which follows then by solving the resulting 1-st order equations. In the process, one explicitly determines the metric on the resolved conifold.

Alternatively, one may start with finding the Ricci flat Kähler metric of the resolved conifold (which, as far as we know, was not previously given in the literature in an explicit form) and then solve the type-IIB supergravity equations for the D3-brane ansatz with constant dilaton and 3-form fluxes representing the inclusion of fractional D3-branes. As in the other two ("standard" and "deformed" conifold) cases, the complex 3-form field turns out to be self-dual. The importance of this property was emphasized in [11, 16] and the $\mathcal{N} = 1$ supersymmetry of such class of backgrounds was recently proved in [17, 18].

In section 2 we shall review the "small resolution" of the conifold [19] and find the corresponding Ricci flat metric explicitly.

In section 3 we shall show how the geometry of $M^{10} = \mathbb{R}^{1,3} \times$(resolved conifold) changes in the presence of D3-branes, i.e. find the analog of the standard D3-brane solution [20, 21] in case when the transverse 6-space is replaced by the resolved

[1]Various aspects of branes on resolved conifold were discussed, e.g. in [12, 13, 14].

conifold (the coefficient in the metric is a harmonic function on the 6-space). In contrast to the D3-brane on the conifold [5] the short distance limit of this supergravity background does not have an AdS_5 factor and is singular. We shall compare this D3-brane solution with the one in the case of the deformed conifold [11]. We consider a radially symmetric solution corresponding to 3-branes smeared over a 2-sphere at the apex. In [13] the 3-branes were instead localized at a point on S^2. The choice of point corresponds to giving expectation values for scalar fields in the dual field theory, breaking gauge symmetry and conformal invariance. The solution in [13] was non-singular in IR, approaching $AdS_5 \times S^5$. It seems that the averaging over S^2 causes a singularity (present also in analogous D3-brane solution with deformed conifold as transverse space).[2]

In section 4 we shall generalize the D3-brane solution of section 3 to the presence of fractional D3-branes on the resolved conifold. We shall analyze the limits of the solution and show that it has a short-distance singularity. This is a repulson-type singularity, so one may hope that it may be resolved by the mechanism of [22].

In section 5 we shall explain how the same solution can be obtained from a 1-d action for radial evolution admitting a superpotential, i.e. by solving a system of 1-st order equations as in [10]. We shall point out that a similar superpotential exists also in the deformed conifold case of [11]. As we shall demonstrate in the process, making simple ansatze for the 6-d part of the metric and identifying the 1-st order systems associated with the Ricci-flatness equations allows one to find the explicit forms of the resolved and deformed conifold metrics in a straightforward way. Identifying explicitly the 1-st order system (whose existence is expected on the grounds of residual supersymmetry) is useful for generalizations and for establishing a potential correspondence between the radial evolution on the supergravity side and the $\mathcal{N} = 1$ supersymmetric RG flow in the dual gauge theory.

2. Metric of resolved conifold

The purpose of this section is to write down explicitly the Ricci flat metric on the resolved conifold following the detailed discussion in [19]. Though it is not possible to introduce a globally well-defined metric on the resolved conifold, one can find a metric on each of the two covering patches. The conifold can be described by the following quadric in $\mathbf{C}^4$: $\sum_{i=1}^4 w_i^2 = 0$.[3] This equation can be written as

$$\det \mathcal{W} = 0\,, \qquad \text{i.e.} \qquad XY - UV = 0\,,$$
$$\mathcal{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} w_3 + iw_4 & w_1 - iw_2 \\ w_1 + iw_2 & -w_3 + iw_4 \end{pmatrix} \equiv \begin{pmatrix} X & U \\ V & Y \end{pmatrix}. \qquad (2.1)$$

[2] We are grateful to I. Klebanov for this suggestion and explaining the relation to [13].

[3] More details on the topological structure of the resolved conifold as a $\mathbb{C}^2$ bundle over $\mathbb{CP}^1$ can be found in [19].

The resolution of the conifold can be naturally described in terms of (X, Y, U, V). Resolving the conifold means replacing the equation $XY - UV = 0$ by the pair of equations

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0\,, \tag{2.2}$$

where $\lambda_1\lambda_2 \neq 0$. Note that $(\lambda_1, \lambda_2) \in \mathbb{CP}^1$ (any pair obtained from a given one by multiplication by a nonzero complex number is also a solution). Thus (λ_1, λ_2) is uniquely characterized by $\lambda = \lambda_2/\lambda_1$ in the region where $\lambda_1 \neq 0$. Working on this patch a solution to (2.2) takes the form[4]

$$\mathcal{W} = \begin{pmatrix} -U\lambda & U \\ -Y\lambda & Y \end{pmatrix}. \tag{2.3}$$

Thus (U, Y, λ) are the three complex coodinates characterizing the resolved conifold in the patch where $\lambda_1 \neq 0$.

The conifold metric is $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} \mathrm{K}$, where K is the Kähler potential. In contrast to the the cases of the conifold or the deformed conifold, here the Kähler potential is not a globally defined quantity, and is not a function of only the radial coordinate defined by

$$r^2 = \mathrm{tr}(\mathcal{W}^\dagger \mathcal{W}) = (1 + |\lambda|^2)(|U|^2 + |Y|^2)\,. \tag{2.4}$$

Following the analysis of [19], based on the transformation of the coordinates in the overlap region, one concludes that the most general Kähler potential is of the form

$$\mathrm{K} = F(r^2) + 4a^2 \ln(1 + |\lambda|^2)\,, \tag{2.5}$$

where F is a function of r^2 and a is the "resolution" parameter ($a = 0$ is the conifold case). Thus the metric is

$$ds^2 = F' \,\mathrm{tr}(d\mathcal{W}^\dagger d\mathcal{W}) + F'' |\,\mathrm{tr}(\mathcal{W}^\dagger d\mathcal{W})|^2 + 4a^2 \frac{|d\lambda|^2}{(1+|\lambda|^2)^2}\,, \qquad F' \equiv \frac{dF}{dr^2}\,. \tag{2.6}$$

The Ricci tensor for a Kähler metric is $R_{m\bar{n}} = -\partial_m \partial_{\bar{n}} \ln \det g_{m\bar{n}}$, where for the metric in (2.6)

$$\det g_{m\bar{n}} = F'(F' + r^2 F'')(4a^2 + r^2 F')\,. \tag{2.7}$$

The Ricci-flatness condition implies

$$\gamma'\gamma(\gamma + 4a^2) = \frac{2}{3} r^2\,, \qquad \gamma \equiv r^2 F'\,, \qquad \gamma' \equiv \frac{d\gamma}{dr^2}\,, \tag{2.8}$$

which is integrated to give

$$\gamma^3 + 6a^2\gamma^2 - r^4 = 0\,. \tag{2.9}$$

[4]In the region where λ_1 is allowed to be zero we have $\lambda_2 \neq 0$ and thus the general solution can be written as $\mathcal{W} = \begin{pmatrix} X & -X\mu \\ V & -V\mu \end{pmatrix}$, where $\mu = \lambda_1/\lambda_2$.

We set the integration constant to zero, assuming that $\gamma(0) = 0$ (as should be true in the $a = 0$ case of the conifold).[5] The real solution is[6]

$$\gamma = -2a^2 + 4a^4 N^{-1/3}(r) + N^{1/3}(r), \qquad N(r) \equiv \frac{1}{2}(r^4 - 16a^6 + \sqrt{r^8 - 32a^6 r^4}) \,. \tag{2.10}$$

In the conifold case $a = 0$ we have $\gamma = r^{4/3}$ [19]. Note also that

$$\gamma(r \to 0) = \frac{1}{\sqrt{6}a} r^2 - \frac{1}{72a^4} r^4 + O(r^6) \,, \qquad \gamma(r \to \infty) = r^{4/3} - 2a^2 + O(r^{-4/3}) \,. \tag{2.11}$$

To write down the resolved conifold metric explicitly we will parametrize $\mathcal{W}$ in terms of the two sets of Euler angles, exploiting the fact that the resolved conifold solution for $\mathcal{W}$ has $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry[7]

$$\begin{aligned} U &= r e^{i/2(\psi + \phi_1 + \phi_2)} \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \,, \qquad Y = r e^{\frac{i}{2}(\psi - \phi_1 + \phi_2)} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \,, \\ \lambda &= e^{-i\phi_2} \tan\frac{\theta_2}{2} \,. \end{aligned} \tag{2.12}$$

Then the resolved conifold metric takes the form

$$\begin{aligned} ds_6^2 = \gamma' dr^2 + \frac{1}{4}\gamma \sum_{i=1}^{2} \left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) + \frac{1}{4}\gamma' r^2 \left(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i\right)^2 + \\ + a^2 (d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \,. \end{aligned} \tag{2.13}$$

Note that the parameter a introduces asymmetry between the two spheres. Defining the veilbeins

$$e_\psi = d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i \,, \qquad e_{\theta_i} = d\theta_i \,, \qquad e_{\phi_i} = \sin\theta_i d\phi_i \,, \qquad i = 1, 2 \,, \tag{2.14}$$

the metric can be written as

$$ds_6^2 = \gamma' dr^2 + \frac{1}{4}\gamma' r^2 e_\psi^2 + \frac{1}{4}\gamma \left(e_{\theta_1}^2 + e_{\phi_1}^2\right) + \frac{1}{4}(\gamma + 4a^2)\left(e_{\theta_2}^2 + e_{\phi_2}^2\right) . \tag{2.15}$$

As follows from (2.11), for small r the S^3 (ψ, θ_1, ϕ_1) part of the metric shrinks to zero size while the S^2 (θ_2, ϕ_2) part stays finite with radius a.

Since $\gamma' d(r^2) = d\gamma$ and $\gamma' = \frac{2r^2}{3\gamma(\gamma + 4a^2)}$ it is very convenient to consider γ as a new radial coordinate introducing

$$\rho^2 \equiv \frac{3}{2}\gamma \,. \tag{2.16}$$

[5]One could keep the integration constant in (2.9) which would lead to more general geometries. These more general geometries are compatible with metrics on the conifold that are not the cone over $\mathbb{T}^{1,1}$ but are nevertheless Kähler, Ricci-flat metrics.

[6]This expression applies for all $r^2 > 0$ provided for $r^2 < 4\sqrt{2}a^3$ one uses the cubic root $(-1)^{1/3} = \frac{1+i\sqrt{3}}{2}$ (while N becomes complex, γ stays real).

[7]Here $\psi = \psi_1 + \psi_2$, and $(\theta_1, \phi_1, \psi_1)$ and $(\theta_2, \phi_2, \psi_2)$ correspond to the two SU(2)'s.

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This allows one to avoid the issue of how to define the expression (2.10) in different regions. Then using (2.8) and (2.9) the resolved conifold metric can be written simply as

$$ds_6^2 = \kappa^{-1}(\rho)d\rho^2 + \frac{1}{9}\kappa(\rho)\rho^2 e_\psi^2 + \frac{1}{6}\rho^2\left(e_{\theta_1}^2 + e_{\phi_1}^2\right) + \frac{1}{6}(\rho^2+6a^2)\left(e_{\theta_2}^2 + e_{\phi_2}^2\right), \quad (2.17)$$

where

$$\kappa(\rho) \equiv \frac{\rho^2+9a^2}{\rho^2+6a^2}. \quad (2.18)$$

This is the explicit SU(2)×SU(2) invariant form of the resolved conifold metric which we shall use in what follows.[8] When the resolution parameter a goes to zero or when $\rho \to \infty$ it reduces to the standard conifold metric with $\mathbb{T}^{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ as the base [19, 23]

$$(ds_6^2)_{\rho\to\infty} = d\rho^2 + \rho^2\left[\frac{1}{9}e_\psi^2 + \frac{1}{6}\left(e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2\right)\right]. \quad (2.19)$$

For small ρ the metric (2.17) reduces to

$$(ds_6^2)_{\rho\to 0} = \frac{2}{3}d\rho^2 + \frac{1}{6}\rho^2\left(e_\psi^2 + e_{\theta_1}^2 + e_{\phi_1}^2\right) + a^2\left(e_{\theta_2}^2 + e_{\phi_2}^2\right). \quad (2.20)$$

This shows once again that near the apex ($\rho = 0$) the S^3 part shrinks to zero size while the radius of S^2 (θ_2, ϕ_2) part approaches finite value equal to a.

3. D3-branes on resolved conifold

As is well known, given a Ricci flat 6-d space with the metric g_{mn} one can construct the following generalization of the standard [20, 21] brane solution (see, e.g. [4, 24, 25])

$$ds_{10}^2 = h^{-1/2}(y)dx^\mu dx^\mu + h^{1/2}(y)g_{mn}(y)dy^m dy^n, \quad (3.1)$$

$$F_5 = (1+*)dh^{-1}\wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \qquad \Phi = \text{const}, \quad (3.2)$$

where h is a harmonic function on the transverse 6-d space:

$$\frac{1}{\sqrt{g}}\partial_m\left(\sqrt{g}g^{mn}\partial_n h\right) = 0. \quad (3.3)$$

Let us solve (3.3) for the resolved conifold metric (2.17) assuming $h = h(\rho)$. Using that $\sqrt{g} = \frac{1}{108}\rho^3(\rho^2+6a^2)\sin\theta_1\sin\theta_2$ we get

$$h = h_0 + \frac{2L^4}{9a^2\rho^2} - \frac{2L^4}{81a^4}\ln\left(1+\frac{9a^2}{\rho^2}\right), \quad (3.4)$$

[8] It is easy to check directly that this metric is indeed Ricci flat.

where we have chosen the integration constant so that in the $a \to 0$ (or, equivalently, large ρ) limit the solution approaches the standard flat space or conifold one

$$h(\rho \to \infty) = h_0 + \frac{L^4}{\rho^4} \,. \tag{3.5}$$

For small values of the radius ρ we get[9]

$$h(\rho \to 0) = \frac{b^2}{\rho^2} \,, \qquad b^2 = \frac{2L^4}{9a^2} \,, \tag{3.6}$$

so that the 10-d metric becomes

$$\begin{aligned} (ds_{10}^2)_{\rho\to 0} &= \frac{\rho}{b} dx^\mu dx^\mu + \frac{b}{\rho} (ds_6^2)_{\rho \to 0} \\ &= b \left[\frac{y^2}{b^2} dx^\mu dx^\mu + \frac{8}{3} dy^2 + \frac{1}{6} y^2 \left(e_\psi^2 + e_{\theta_1}^2 + e_{\phi_1}^2 \right) + a^2 y^2 \left(e_{\theta_2}^2 + e_{\phi_2}^2 \right) \right] , \end{aligned} \tag{3.7}$$

where $y = \sqrt{\rho}$ and we used the expression (2.20) for the short-distance limit of the resolved conifold metric.

The S^3 part of the 10-d metric still shrinks to zero size but the size of S^2 rather than approaching the constant a now blows up at $\rho = 0$. It is easy to check that the point $\rho = 0$ is the curvature singularity (while the Ricci scalar vanishes as for any D3-brane solution of the type (3.2), the Ricci tensor is singular). This behaviour is to be compared with one in the case of the D3-branes at the conifold singularity where the short-distance limit of the geometry was regular $AdS_5 \times \mathbb{T}^{1,1}$ space (see [5]).

For completeness, let us compare the above solution with the one in the case when the transverse 6-space is the deformed conifold with the metric [11, 19, 24, 26]

$$ds_6^2 = \frac{1}{2} \epsilon^{4/3} \mathcal{K} \left[(3\mathcal{K}^3)^{-1} (d\tau^2 + g_5^2) + \sinh^2 \frac{\tau}{2} \, (g_1^2 + g_2^2) + \cosh^2 \frac{\tau}{2} \, (g_3^2 + g_4^2) \right] , \tag{3.8}$$

where ϵ is the deformation parameter,

$$\mathcal{K}(\tau) = \frac{\left[\frac{1}{2} \sinh(2\tau) - \tau \right]^{1/3}}{\sinh \tau} \,, \tag{3.9}$$

and the 1-forms g_n defined in [11] are

$$\begin{aligned} g_1 &= -\frac{\epsilon_2 + e_{\phi_1}}{\sqrt{2}} \,, \qquad g_2 = -\frac{\epsilon_1 - e_{\theta_1}}{\sqrt{2}} \,, \qquad g_3 = \frac{\epsilon_2 - e_{\phi_1}}{\sqrt{2}} \,, \qquad g_4 = \frac{\epsilon_1 + e_{\theta_1}}{\sqrt{2}} \,, \\ g_5 &= e_\psi \,, \\ \epsilon_1 &\equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2 \,, \qquad \epsilon_2 \equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2 \,. \end{aligned} \tag{3.10}$$

[9] It is easy to check that h does not vanish at any real value of r. Introducing $x = 9a^2/\rho^2$ and $c^2 = \frac{81 a^4 h_0}{2L^4}$ the equation $h(r) = 0$ becomes $\ln(1+x) - x = c^2$ which has no $x > 0$ solutions.

The harmonic function h in (3.1) is then found to be

$$h(\tau) = h_0 - h_1 \int \frac{d\tau}{\left[\frac{1}{2}\sinh(2\tau) - \tau\right]^{2/3}} = \begin{cases} 1 + \frac{3}{4^{2/3}} h_1 e^{-4\tau/3}\,, & \tau \to \infty \\ \left(\frac{3}{4}\right)^{2/3} h_1 \tau^{-1}\,, & \tau \to 0\,. \end{cases} \tag{3.11}$$

Introducing $\rho \sim e^{\tau/3}$ for large τ we recover the D3-brane on the conifold limit with $h = h_0 + L^4/\rho^4$. For small values of τ we get

$$\begin{aligned} ds^2_{10} &= \frac{\sqrt{\rho}}{m}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{m}{\sqrt{\rho}}(ds^2_6)_{\rho\to 0}\,, \\ (ds^2_6)_{\rho\to 0} &= d\rho^2 + \frac{1}{2}\rho^2 d\Omega^2_2 + \frac{\epsilon^{4/3}}{(12)^{1/3}}d\Omega^2_3\,, \end{aligned} \tag{3.12}$$

where $\rho \equiv \frac{\epsilon^{2/3}}{2^{5/6}3^{1/6}}\tau$, and $m^2 = L^4\epsilon^{-2}(3/2)^{-5/2}$. In the short distance limit of the 6-d deformed conifold metric the 2-sphere shrinks to zero size while the 3-sphere part has finite radius related to the deformation parameter ϵ. In the 10-d metric the S^2 part still shrinks to zero size but the radius of the S^3 part blows up at the point $\rho = 0$ which is the curvature singularity. As in the resolved conifold case, the near-core geometry is singular. That is why to get a regular solution after adding fractional D3-branes [11] one needs to set the "bare" D3-brane charge to zero to make possible for the 5-form field (and thus for the Ricci tensor) to vanish at small ρ.

4. Fractional D3-branes on resolved conifold

Let us now study a generalization of the D3-brane solution of the previous section to the case of additional 3-form fluxes, with the aim to find the analog of the solution of [10] describing a collection of regular and fractional D3-branes on the conifold in the case when the conifold is replaced by the resolved conifold. The ansatz for the metric will be the same as in (3.1),

$$ds^2_{10} = h^{-1/2}(\rho)dx^\mu dx^\mu + h^{1/2}(\rho)ds^2_6\,, \tag{4.1}$$

where ds^2_6 will be the metric of the resolved conifold (2.17). Our ansatz for the NS-NS 2-form will be a natural generalization of the ansatz in [10] motivated by an asymmetry between the two S^2 parts in the resolved conifold metric (2.17)

$$\begin{aligned} B_2 &= f_1(\rho)e_{\theta_1}\wedge e_{\phi_1} + f_2(\rho)e_{\theta_2}\wedge e_{\phi_2}\,, \\ H_3 &= dB_2 = d\rho\wedge\left[f_1'(\rho)e_{\theta_1}\wedge e_{\phi_1} + f_2'(\rho)e_{\theta_2}\wedge e_{\phi_2}\right]. \end{aligned} \tag{4.2}$$

The "conifold" ansatz [8, 10] corresponds to $f_1 = -f_2$. The ansatz for the R-R 3-form F_3 is dictated by the closure condition $dF_3 = 0$, i.e. the forms F_3 and F_5 will

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be taken in the same form as in [8, 10][10]

$$F_3 = Pe_\psi \wedge (e_{\theta_2} \wedge e_{\phi_2} - e_{\theta_1} \wedge e_{\phi_1})\,,$$
$$F_5 = \mathcal{F} + *\mathcal{F}\,, \qquad \mathcal{F} = K(\rho) e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2}\,. \tag{4.3}$$

Then using the metric (2.17) the 10-d duals of these forms are found to be

$$*\mathcal{F} = \frac{108K}{\rho^3(\rho^2+9a^2)h^2} d\rho \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\,,$$
$$*F_3 = \frac{3P\rho}{(\rho^2+9a^2)h} d\rho \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \left(e_{\theta_1} \wedge e_{\phi_1} - \Gamma^2 e_{\theta_2} \wedge e_{\phi_2}\right)\,,$$
$$*H_3 = -\frac{\rho^2+9a^2}{3\rho h} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge e_\psi \left(f_1' e_{\theta_2} \wedge e_{\phi_2} + \Gamma^{-2} f_2' e_{\theta_1} \wedge e_{\phi_1}\right)\,. \tag{4.4}$$

Here

$$\Gamma \equiv \frac{\rho^2+6a^2}{\rho^2}\,. \tag{4.5}$$

is the ratio of the squares of the radii of the two spheres in the metric (2.17) and its difference from 1 is a signature of the resolution ($a \neq 0$).

As in [10, 11] we shall assume that the dilaton Φ is constant. Then the F_3 equation of motion $d(e^\Phi * F_3) = F_5 \wedge H_3$ is satisfied automatically, and from the H_3 equation $d(e^{-\Phi} * H_3) = -F_5 \wedge F_3$ one obtains the following three equations ($e^\Phi = g_s$)

$$\left[\frac{f_1'(\rho^2+9a^2)}{h\rho}\right]' = \frac{324 g_s P K}{h^2\rho^3(\rho^2+9a^2)}\,,$$
$$\left[\frac{f_2'(\rho^2+9a^2)}{h\rho\Gamma^2}\right]' = -\frac{324 g_s P K}{h^2\rho^3(\rho^2+9a^2)}\,, \tag{4.6}$$
$$f_1' + \Gamma^{-2} f_2' = 0\,. \tag{4.7}$$

It follows from (4.7) that for $\Gamma = 1$ one should have $f_2 = -f_1$ (modulo an irrelevant constant) which was precisely the assumption of [10] in the $a = 0$ case.

The constant dilaton condition implies $H_3^2 = e^{2\Phi} F_3^2$, i.e. using (2.17) we get[11]

$$f_1'^2 + \Gamma^{-2} f_2'^2 = \frac{9 g_s^2 P^2}{k^2 \rho^2}\left(1 + \Gamma^{-2}\right)\,. \tag{4.8}$$

Combined with (4.7) that gives

$$f_1' = 3 g_s P \frac{\rho}{\rho^2+9a^2}\,, \qquad f_2' = -3 g_s P \frac{(\rho^2+6a^2)^2}{\rho^3(\rho^2+9a^2)}\,. \tag{4.9}$$

[10] Note that our definition of the basis of 1-forms differ from [10] by numerical factors, so that the constant P and function K here are related to the ones in [10] by $P \to \frac{1}{18\sqrt{2}} P$, $K \to \frac{1}{108} K$. Also, in the case of [10] $f_1 = -f_2 = \frac{1}{6\sqrt{2}} T$.

[11] As in [10, 11], the axion equation is satisfied automatically since $H_3 \cdot F_3 = 0$.

It is easy to see from the above relations that, as in the conifold [10] and the deformed conifold cases [11], the forms H_3 and F_3 are dual to each other in the 6-d sense. This property, together with the Calabi-Yau nature of the (original, deformed or resolved) conifold metrics implies the $\mathcal{N} = 1, d = 4$ supersymmetry of the resulting backgrounds [17, 18].

The Bianchi identity for the 5-form $d * F_5 = dF_5 = H_3 \wedge F_3$ gives

$$K' = P(f_1' - f_2') , \qquad \text{i.e.} \qquad K = Q + P(f_1 - f_2) . \tag{4.10}$$

The symmetries of the metric ansatz imply (again, as in the other two conifold cases [10, 11]) that to determine the function $h(\rho)$ it is sufficient to consider the trace of the Einstein equations, $R = -\frac{1}{2}\Delta h = \frac{1}{24}(e^{-\Phi} H_3^2 + e^{\Phi} F_3^2)$, i.e.

$$h^{-3/2} \frac{1}{\sqrt{g}} \partial_\rho \left(\sqrt{g} g^{\rho\rho} \partial_\rho h\right) = -\frac{1}{12}(g_s^{-1} H_3^2 + g_s F_3^2) = -\frac{1}{6} g_s F_3^2 , \tag{4.11}$$

where g_{mn} is the 6-d metric (2.17) ($g^{\rho\rho} = \kappa(\rho) = \frac{\rho^2+9a^2}{\rho^2+6a^2}$, $\sqrt{g} \sim \rho^3(\rho^2 + 6a^2)$), i.e.

$$\left[\rho^3(\rho^2 + 9a^2) h'\right]' = -324 g_s P^2 \frac{\rho(1 + \Gamma^2)}{\rho^2 + 9a^2} . \tag{4.12}$$

Integrating this equation we get

$$h' = -\frac{36 g_s P^2}{\rho^3(\rho^2 + 9a^2)} \left(3Q - \frac{18a^2}{\rho^2} + \ln[\rho^8(\rho^2 + 9a^2)^5]\right) , \tag{4.13}$$

where we have chosen the integration constant to be related to the one in (4.10). From (4.9) we find (we omit trivial constants of integration)

$$\begin{aligned} f_1(\rho) &= \frac{3}{2} g_s P \ln(\rho^2 + 9a^2) , \\ f_2(\rho) &= \frac{1}{6} g_s P \left(\frac{36a^2}{\rho^2} - \ln[\rho^{16}(\rho^2 + 9a^2)]\right) , \end{aligned} \tag{4.14}$$

and thus from (4.10)

$$K(\rho) = Q - \frac{1}{3} g_s P^2 \left(\frac{18a^2}{\rho^2} - \ln\left[\rho^8(\rho^2 + 9a^2)^5\right]\right) . \tag{4.15}$$

Note that (4.13) and (4.15) imply that

$$h' = -108 \rho^{-3} (\rho^2 + 9a^2)^{-1} K(\rho) . \tag{4.16}$$

Integrating (4.13) one can find the explicit form of $h(\rho)$ which is not very illuminating as it contains the special function $Li_2(-\rho/3a)$. The constants Q and P are proportional to the numbers N and M of regular and fractional D3-branes.

In the large ρ ($\rho \gg 3a$) limit we reproduce the solution of [10] with its characteristic logarithmic behavior

$$
\begin{aligned}
&f_1' = 3g_sP\rho^{-1}\,, \qquad f_2' = -3g_sP\rho^{-1}\,, \qquad K = Q + 6g_sP^2\ln\rho\,,\\
&h = h_0 + \frac{L^4 + 162g_sP^2(\ln\rho + 1/4)}{\rho^4}\,,
\end{aligned}
\tag{4.17}
$$

where $L^4 = 27Q$ (and $h_0 = g_s^{-1}$ as we use the Einstein-frame metric).

In the short distance limit ($\rho \ll 3a$) the solution becomes

$$
\begin{aligned}
&f_1' = \frac{g_sP}{3a^2}\rho\,, \qquad f_2' = -\frac{12g_sPa^2}{\rho^3}\,, \qquad K = Q - \frac{6g_sP^2a^2}{\rho^2}\,,\\
&h = h_0 + \frac{6Q}{a^2\rho^2} - \frac{18g_sP^2}{\rho^4}\,.
\end{aligned}
\tag{4.18}
$$

The form of h implies the presence of a naked singularity at $\rho = \rho_h$

$$
\rho_h^2 = \frac{3Q}{h_0a^2}\left(\sqrt{1 + 2h_0g_sP^2a^4Q^{-2}} - 1\right). \tag{4.19}
$$

For small number of fractional D3-branes ($P \ll Q$) the singularity is located at $\rho_h^2 = 3g_sP^2Q^{-1}a^2$.[12] At the same time, the five-form coefficient $K(\rho)$ (4.15) vanishes at $\rho = \rho_K,\ \ \rho_K = \sqrt{2}\rho_h > \rho_h$.

One may expect that this naked singularity may be resolved by the enhançon mechanism [22] (as was originally expected [10] for the singularity in the conifold case). First, the singularity is of the right repulson type [27]. Second important feature is the underlying SU(2) symmetry of the 6-d part of the metric (see [22, footnote 2]). To make the argument for such resolution at a quantitative level is, however, non trivial.[13]

If a mechanism similar to the one in [22] does apply in the present case, then the geometry should "stop" at $\rho = \rho_K$ before reaching the singularity at ρ_h. Expanding around $\rho = \rho_K$ we get

$$
K(\rho_K + \tilde\rho) = \frac{2Q}{\rho_K}\tilde\rho + O(\tilde\rho^2)\,, \qquad h(\rho_K + \tilde\rho) = h_0 + \frac{Q^2}{2g_sP^2a^4} + O(\tilde\rho^2)\,. \tag{4.20}
$$

This is similar to the IR behavior found in the deformed conifold case [11]. In particular, the constant value of the warp factor h at short distances should imply again confinement in the IR.

[12] One obtains the same value by simply sending $h_0 \to 0$, as naively expected in the limit of small radius.

[13] Reference [22] used the form of the effective action for the D6-branes wrapped over on K3 that probe the geometry. The case of D6 on K3 is similar to the case of D4 on K3 which has been extensively discussed in the literature [28]. In the present case we are dealing with a Calabi-Yau of dimension 6 and the D5-brane we are dealing with here is wrapping a two-cycle rather than the whole space. Thus we have non-trivial tangent and normal bundles which will affect the Chern-Simons term. The Dirac-Born-Infeld part of the action is also different (see [29] for details).

5. Superpotential and first order system

Let us now demonstrate how the 1-st order system of equations and the solution of the previous section can be derived directly without using the expression for the resolved conifold metric. We shall follow the original approach of [10], i.e. start with an ansatz for the 10-d metric and p-form fields which has the required symmetries, compute the 1-d action for the radial evolution that reproduces the type-IIB supergravity equations of motion restricted to this ansatz, show that this action admits a superpotential and thus obtain a 1-st order system.

As we shall explain, the same strategy applies also to the case of the deformed conifold ansatz considered in [11]. The corresponding superpotential has essentially the same structure as in the conifold [10] and resolved conifold case, and reproduces the 1-st order system found in [11] thus checking its consistency.

5.1 Resolved conifold case

Let us choose the 10-d metric in the following "5+5" form

$$ds_{10}^2 = e^{2p-x}(e^{2A}dx^\mu dx^\mu + du^2) + \left[e^{-6p-x}e_\psi^2 + e^{x+y}(e_{\theta_1}^2 + e_{\phi_1}^2) + \right. \\ \left. + e^{x-y}(e_{\theta_2}^2 + e_{\phi_2}^2)\right] , \tag{5.1}$$

where A, p, x, y are functions of a radial coordinate u. Note that the metric of the previous section (4.1) and (2.17) belongs to this class (u is related to ρ). To be able to describe the resolved conifold case we have included the function y which measures an "asymmetry" between the two S^2 parts (y was set to zero in the "symmetric" conifold ansatz [10]). The ansatz for the remaining fields will be the same as in (4.2) and (4.3), i.e.[14]

$$H_3 = du \wedge [f_1'(u)e_{\theta_1} \wedge e_{\phi_1} + f_2'(u)e_{\theta_2} \wedge e_{\phi_2}] , \tag{5.2}$$

$$F_3 = Pe_\psi \wedge (e_{\theta_2} \wedge e_{\phi_2} - e_{\theta_1} \wedge e_{\phi_1}) , \tag{5.3}$$

$$F_5 = \mathcal{F} + *\mathcal{F} , \qquad \mathcal{F} = K(u)e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2} , \tag{5.4}$$

$$K(u) \equiv Q + P[f_1(u) - f_2(u)] . \tag{5.5}$$

Here we have explicitly used the constraint (4.10) following from the Bianchi identity for the 5-form field (so that the Bianchi identities for all three p-form fields are satisfied). Thus only f_1 and f_2 will be considered as independent functions of u coming out of the p-form part (5.2)–(5.5) of the ansatz. We shall assume that the axion is zero (this is consistent with (5.3)) but will keep the dilaton $\Phi = \Phi(u)$.

[14] In this section prime will denote derivatives over u.

The type-IIB supergravity equations of motion follow from the action

$$S_{10} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x \left(\sqrt{-g_{10}}\left[R_{10} - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{12}e^{-\Phi}(\partial B_2)^2 - \frac{1}{2}e^{2\Phi}(\partial\mathcal{C})^2 - \right.\right.$$
$$\left. - \frac{1}{12}e^{\Phi}(\partial C_2 - \mathcal{C}\partial B_2)^2 - \frac{1}{4\cdot 5!}F_5^2\right] -$$
$$\left. - \frac{1}{2\cdot 4!\cdot(3!)^2}\epsilon_{10}C_4\partial C_2\partial B_2 + \cdots\right), \tag{5.6}$$
$$(\partial B_2)_{\dots} = 3\partial_{[.}B_{..]}\,, \qquad (\partial C_4)_{\dots} \equiv 5\partial_{[.}C_{\dots]}\,, \qquad F_5 = \partial C_4 + 5(B_2\partial C_2 - C_2\partial B_2)\,,$$

supplemented with the on-shell constraint $F_5 = *F_5$ [33]. The 1-d action reproducing the resulting equations of motion restricted to the above ansatz has the following general structure

$$S = c\int du\; e^{4A}\left[3A'^2 - \frac{1}{2}G_{ab}(\varphi)\varphi'^a\varphi'^b - V(\varphi)\right], \tag{5.7}$$

where $c = -4\frac{Vol_9}{2\kappa_{10}^2}$. It should be supplemented with the "zero-energy" constraint

$$3A'^2 - \frac{1}{2}G_{ab}(\varphi)\varphi'^a\varphi'^b + V(\varphi) = 0\,. \tag{5.8}$$

The existence of a superpotential (usually associated with residual supersymmetry, see, e.g. [30, 31] but also [32]) means that V in (5.7) can be represented in the form

$$V = \frac{1}{8}G^{ab}\frac{\partial W}{\partial\varphi^a}\frac{\partial W}{\partial\varphi^b} - \frac{1}{3}W^2\,. \tag{5.9}$$

In this case the 2-nd order equations following from (5.7) and the constraint (5.8) are satisfied on the solutions of the 1-st order system

$$\varphi'^a = \frac{1}{2}G^{ab}\frac{\partial W}{\partial\varphi^b}\,, \qquad A' = -\frac{1}{3}W(\varphi)\,. \tag{5.10}$$

In our present case we have 6 dynamical variables $\varphi^a = (x, y, p, \Phi, f_1, f_2)$. As follows from (5.6) in the case of the ansatz (5.1)–(5.4)

$$G_{ab}(\varphi)\varphi'^a\varphi'^b = x'^2 + \frac{1}{2}y'^2 + 6p'^2 + \frac{1}{4}\left[\Phi'^2 + P^2e^{-\Phi-2x}(e^{-2y}f_1'^2 + e^{2y}f_2'^2)\right], \tag{5.11}$$
$$V(\varphi) = \frac{1}{4}e^{-4p-4x}\cosh 2y - e^{2p-2x}\cosh y + \frac{1}{8}e^{8p}\left(2P^2e^{\Phi-2x}\cosh 2y + \right.$$
$$\left. + e^{-4x}K^2\right), \tag{5.12}$$

where we separated the gravity contributions (coming from the R_{10}-term in (5.6)) from the "matter" ones and it is assumed that K is a combination of f_1, f_2 in (5.5).

Similar expressions corresponding to the case of

$$y = 0\,, \qquad f_1' = -f_2' \tag{5.13}$$

appeared in [10]. Indeed, that restriction was consistent. As follows from (5.7), (5.11) and (5.12) the equation for y is satisfied by $y = 0$ if $f_1'^2 = f_2'^2$. Also, the potential (5.12) depends only on one of the two combinations $f_\pm \equiv f_1 \pm f_2$, so that the equation for f_+ is satisfied automatically by $f_+' = 0$ if $y = 0$. The 1-d action of [10] may be found by eliminating f_+ from the action using its equation of motion and then setting $y = 0$.

It is quite remarkable that, just like in the "symmetric" case considered in [10], the more general system (5.11) and (5.12) still admits a simple superpotential W given by the direct superposition of the gravitational and matter parts

$$\begin{aligned} W(\varphi) &= e^{4p} + e^{-2p-2x}\cosh y + \frac{1}{2}e^{4p-2x}K \\ &= e^{4p} + e^{-2p-2x}\cosh y + \frac{1}{2}e^{4p-2x}[Q + P(f_1 - f_2)] \,. \end{aligned} \tag{5.14}$$

Note that the dilaton factors in the kinetic (5.11) and potential (5.12) terms conspire so that the superpotential does not depend on the dilaton. This implies that $\Phi = \text{const}$ on the solution of the resulting 1-st order system of equations (5.10) for $A, x, y, p, \Phi, f_1, f_2$

$$x' = -e^{-2p-2x}\cosh y - \frac{1}{2}e^{4p-2x}K\,, \qquad y' = e^{-2p-2x}\sinh y\,, \tag{5.15}$$

$$p' = \frac{1}{3}e^{4p} - \frac{1}{6}e^{-2p-2x}\cosh y + \frac{1}{6}e^{4p-2x}K\,, \tag{5.16}$$

$$A' = -\frac{1}{3}e^{4p} - \frac{1}{3}e^{-2p-2x}\cosh y - \frac{1}{6}e^{4p-2x}K\,, \tag{5.17}$$

$$f_1' = Pe^{\Phi+4p+2y}\,, \qquad f_2' = -Pe^{\Phi+4p-2y}\,, \qquad \Phi' = 0\,. \tag{5.18}$$

We see that (5.13) corresponding to the "standard" conifold case is indeed a special solution of this more general system.

To establish the equivalence of this system with the one found in the previous section, it is useful first to look at the "gravitational sector" equations that do not depend on matter functions f_i. Since the superpotential (5.11) is the direct sum of the gravitational and matter terms, $M^{10} = \mathbb{R}^4\times$ (resolved conifold) should be a solution to these equations with $K = 0$. Indeed, as follows from (5.15)–(5.17) the factor $e^{2p-x+2A}$ that multiplies $\mathbb{R}^4$ part in (5.1) satisfies

$$h' = -Khe^{4p-2x}\,, \qquad h^{-1/2} \equiv e^{2p-x+2A}\,, \tag{5.19}$$

so that h (which at the end should be the same as in (4.1)) is constant if K is set equal to 0. The equations for x, y, p with $K = 0$ imply

$$\begin{aligned} &\frac{dx}{dy} = -\coth y\,, \qquad e^{2x} = b^2\sinh^{-2} y\,, \\ &\frac{dq}{dy} = 2b^3(\sinh y)^{-4}e^q\,, \qquad e^{-q} = b^3\Big(\frac{\cosh y - \frac{1}{3}\cosh 3y}{\sinh^3 y} - c\Big)\,, \\ &q \equiv 6p - x\,. \end{aligned} \tag{5.20}$$

For a special choice of the integration constants $b = -\frac{1}{2}a^2$, $c = 4/3$ we finally reproduce (using $dy = e^{-2p-2x} \sinh y\, du$ and introducing ρ instead of u to get simple analytic expressions) the resolved conifold metric (2.17) (cf. (5.1))

$$e^{2y} = \frac{\rho^2}{\rho^2 + 6a^2}\,, \qquad e^{2x} = \frac{1}{36}\rho^2(\rho^2 + 6a^2)\,, \qquad e^{-6p+x} = \frac{1}{324}\rho^4(\rho^2 + 9a^2)\,. \quad (5.21)$$

The resolution parameter a is thus one of the three integration constants in the above 1-st order system.

In the presence of matter the full system (5.15)–(5.18) may be solved by first concentrating on the equations that do not involve K, i.e. on the equation for y and for $z = x + 3p$. It is useful to introduce the new radial direction t, $dt = e^{-2p-2x}du$ so that they become $dy/dt = \sinh y$, $dz/dt = -e^{2z} + \frac{3}{2}\cosh y$. One then finds that the ratios of the first and the second, and the second and the third coefficients in the "internal" 5-d part of the metric (5.1), i.e. $e^{-6p-x}/e^{x+y} = e^{-2z-y}$ and $e^{x+y}/e^{x-y} = e^{2y}$, are the same as in the resolved conifold metric, in agreement with (4.1). The rest of the equations then become equivalent (for the special choice of the integration constants) to the system in the previous section.

5.2 Deformed conifold case

Let us now complement the discussion of the deformed conifold case in [11] by demonstrating explicitly that the first order-system there also follows from a simple superpotential which has essentially the same structure as (5.14).

Motivated by the form of the deformed conifold metric (3.8) and (3.10) let us make the following ansatz for the metric (cf. (5.1))

$$ds^2 = e^{2p-x}(e^{2A}dx^\mu dx^\mu + du^2) + \left[e^{-6p-x}g_5^2 + e^{x+y}(g_1^2 + g_2^2) + e^{x-y}(g_3^2 + g_4^2)\right]. \quad (5.22)$$

The ansatz for the p-forms is the same as in [11] (cf. (5.2)–(5.5))

$$\begin{aligned}
H_3 &= du \wedge [f'(u)g_1 \wedge g_2 + k'(u)g_3 \wedge g_4]\,, \\
F_3 &= F(u)g_1 \wedge g_2 \wedge g_5 + [2P - F(u)]g_3 \wedge g_4 \wedge g_5 + \\
&\quad + F'(u)du \wedge (g_1 \wedge g_3 + g_2 \wedge g_4)\,, \\
F_5 &= \mathcal{F}_5 + \mathcal{F}_5^*\,, \qquad \mathcal{F}_5 = K(u)g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5\,, \\
K(u) &\equiv Q + k(u)F(u) + f(u)[2P - F(u)]\,, \qquad (5.23)
\end{aligned}$$

where F, f, k are functions to be determined and P and Q are constants. As in the previous case (5.2), (5.3) and (5.4), we explicitly ensure that the Bianchi identities for the p-forms are satisfied automatically. The independent functions of u which will appear in the 1-d action (5.7) are thus A and $\varphi^a = (x, y, p, \Phi, f, k, F)$. The corresponding kinetic and potential terms in (5.7) are found to be similar to (5.11) and (5.12)

$$
\begin{aligned}
G_{ab}(\varphi)\varphi'^a\varphi'^b &= x'^2 + \frac{1}{2}y'^2 + 6p'^2 + \frac{1}{4}\Big[\Phi'^2 + e^{-\Phi-2x}(e^{-2y}f'^2 + e^{2y}k'^2) + \\
&\qquad\qquad + 2e^{\Phi-2x}F'^2\Big]\,, \\
V(\varphi) &= \frac{1}{4}e^{-4p-4x} - e^{2p-2x}\cosh y + \frac{1}{4}e^{8p}\sinh^2 y + \\
&\quad + \frac{1}{8}e^{8p}\left[\frac{1}{2}e^{-\Phi-2x}(f-k)^2 + e^{\Phi-2x}[e^{-2y}F^2 + e^{2y}(2P-F)^2] + \right. \\
&\qquad \left. + e^{-4x}K^2\right]\,, \qquad (5.24)
\end{aligned}
$$

where K is the combination of the independent functions f, k, F given in (5.23). The corresponding superpotential satisfying (5.9) again does not depend on the dilaton and is a sum of the gravitational and matter parts, i.e. has essentially the same structure as the previous one (5.14)

$$
\begin{aligned}
W(\varphi) &= e^{4p}\cosh y + e^{-2p-2x} + \frac{1}{2}e^{4p-2x}K \\
&= e^{4p}\cosh y + e^{-2p-2x} + \frac{1}{2}e^{4p-2x}[Q + kF + f(2P-F)]\,. \qquad (5.25)
\end{aligned}
$$

Thus there is a close similarity ("duality") between the 1-st order systems for the "resolved" and "deformed" cases.

From (5.10) and (5.25) we find the following set of 1-st order equations for the independent functions $A, x, y, p, f, k, F, \Phi$

$$
\begin{aligned}
x' &= -e^{-2p-2x} - \frac{1}{2}e^{4p-2x}K\,, \qquad y' = e^{4p}\sinh y\,, \\
p' &= \frac{1}{3}e^{4p}\cosh y - \frac{1}{6}e^{-2p-2x} + \frac{1}{6}e^{4p-2x}K\,, \\
A' &= -\frac{1}{3}e^{4p}\cosh y - \frac{1}{3}e^{-2p-2x} - \frac{1}{6}e^{4p-2x}K\,, \\
f' &= e^{\Phi+4p+2y}(2P-F)\,, \qquad k' = e^{\Phi+4p-2y}F\,, \\
F' &= -\frac{1}{2}e^{-\Phi+4p}(f-k)\,, \qquad \Phi' = 0\,. \qquad (5.26)
\end{aligned}
$$

The special solution is $y = 0,\ f = k,\ F = P$. In this case the system (5.26) becomes identical to (5.18)–(5.18) with $f_1 = -f_2 = f$ and both reduce to the "standard" conifold case of [10].

To show that this system contains the solution of [11] we follow the same strategy as in the previous subsection: first analyze the subset of gravitational sector equations to find that the ratios of the functions in the metric are the same as in the case of the deformed conifold and then include the matter part. We again find the relation (5.19) implying $4+6$ factorization of the metric for $K = 0$. The equations for x, y, p

with $K = 0$ here lead to the relations which are very similar ("dual") to (5.20), $dq/dy = 2\coth y$, $e^q = b^2 \sinh^2 y$, $q \equiv 6p - x$, and $dx/dy = -e^{-q} e^{-3x} (\sinh y)^{-1} = -e^{-3x} b^{-2} (\sinh y)^{-3}$. It is useful to introduce the new radial coordinate τ so that

$$\frac{dy}{d\tau} = -\sinh y\,, \qquad e^y = \tanh\frac{\tau}{2}\,, \qquad d\tau \equiv -e^{4p} du\,, \tag{5.27}$$

$$\frac{d(x-6p)}{d\tau} = 2\cosh y\,, \qquad e^{x-6p} = b^2 \sinh^2 \tau\,, \tag{5.28}$$

where b is an integration constant. The remaining equation is

$$\begin{aligned} \frac{dx}{d\tau} &= e^{-6p-2x} = e^{x-6p} e^{-3x} = b^2 \sinh^2 \tau \; e^{-3x}\,, \\ e^{3x} &= c + \frac{3}{2} b^2 \left(\frac{1}{2} \sinh 2\tau - \tau \right). \end{aligned} \tag{5.29}$$

Choosing $c = 0$, $b^2 = \frac{1}{96}\epsilon^4$ we thus reproduce the deformed conifold metric (3.8)

$$\begin{aligned} e^{-6p-x} &= e^{2p-x} e^{-8p} = \frac{1}{6}\epsilon^{4/3} \mathcal{K}^{-2}\,, \qquad e^{x+y} = \frac{1}{2}\epsilon^{4/3} \mathcal{K} \sinh^2\frac{\tau}{2}\,, \\ e^{x-y} &= \frac{1}{2}\epsilon^{4/3} \mathcal{K} \cosh^2 \frac{\tau}{2}\,, \end{aligned} \tag{5.30}$$

where $\mathcal{K}(\tau)$ was defined in (3.9) and ϵ is the deformation parameter.[15]

It is quite remarkable that making simple ansatze (5.1) or (5.22) for the 6-d part of the metric one finds that the 1-d action leading to the associated Einstein (Ricci-flatness) equations admit a superpotential, and that the solutions of the corresponding 1-st order systems are the resolved (2.17) and the deformed (3.8) conifold metrics respectively!

In the general case the system (5.26) can be solved by starting with the equations that do not involve matter functions: equation for y (5.27) and the following combination of the equations for x and p:

$$\frac{d(3p+x)}{d\tau} = -\cosh y + \frac{3}{2} e^{-2(3p+x)}\,. \tag{5.31}$$

This equation is solved by first introducing $w = 3p + x + \ln\sinh\tau$. As a result,

$$e^{3p+x} = \sqrt{\frac{3}{2}} (\sinh\tau)^{-1} \left(\frac{1}{2}\sinh 2\tau - \tau \right)^{1/2}\,, \tag{5.32}$$

where we set the integration constant to zero so that e^{3p+x} is exactly the same as in (5.30).

[15] Here and in the previous subsection we make a specific choice of the integration constants in order to match the standard metrics on the conifolds. Keeping the integration constants arbitrary produces a more general class of metrics. The existence of these more general metrics was mentioned also in section 2.

This implies that as in the resolved conifold case of the previous subsection, the ratios of the coefficients in the internal 5-d part of the metric (5.22), i.e. $e^{-6p-x}/e^{x+y} = e^{-2(3p+x)-y}$ and $e^{x+y}/e^{x-y} = e^{2y}$, have the same values as in the deformed conifold metric (3.8). The solution of the full system is then equivalent to that of [11] for the "D3-brane" ansatz (4.1).

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